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History of Mathematics as an Important Learning Tool: The Case of False Position Method

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Abstract

Purpose: The purpose of this theoretical paper is to present the two false position algorithms, using simple examples, and to suggest ways, that a teacher, exploiting these algorithms, can use to benefit their students in learning. The methods of false position were inventions of ancient civilizations (Egyptians, Babylonians and Chinese). These trial and error methods exploit errors and solve all the problems which are solved algebraically, through the application of first degree equations.

Proposed Conceptual Argument: The findings of many studies support the introduction of new mathematical ideas and concepts through a relevant historical context. In fact, the triptych "History, Mathematics and Education" are key, didactic and methodological axes, able to improve the daily teaching-learning process in current mathematics classes. One case, with rich historical mathematical background which can be used in schools, is the methodological algorithms of simple false position and double false position.

Implications: Through a relevant suggested project, students can get to know Egyptian, Babylonian, Chinese and Arabic Mathematics, as the algorithms of the false position form an apocalyptic episode of the entire history of Mathematics, covering a 4,000 year time period. Also, students can learn that these methods are responsible for the adoption of the symbol "X" for multiplication and the introduction of the signs "-" and "+" in Mathematics. Moreover, the simple starting points of ideas that support the scientific edifice of Mathematics can be understood. Finally, the study of the ratios and proportions can be some attractive teaching parameters and modules of the project.

Keywords: Mathematics, history, school, false position, algorithm.

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Introduction

Mathematics along with Philosophy are the oldest academic disciplines that humankind has known (Krantz, 2010). It is believed (Aczel, 1997; Clawson, 2014; Livio, 2010) that the Pythagoreans were those who coined the words “Philosophy-Φιλοσοφία in Greek” (the search for the nature of things and for the truth of beings but also love for knowledge), as well as “Mathematics-Μαθηματικά in Greek”), meaning anything we have learned. In the Pythagorean School, advanced students were called mathematicians (Perisho, 1965). The word “Mathematics” comes from the ancient Greek word “μάθημα = lesson”, i.e. something that is learned, while through a broad interpretation, the word “Mathematics” can mean, even, study and science.

The history of Mathematics is, rather, different from the history of all other sciences, as Mathematics is much more mystical and can be addressed, perhaps, only to selected groups of insiders (Sarton, 1957). However, it is certain that Mathematics arose out of necessity, when people began to think about the physical world or about the world of ideas contained in laws and, even, in Theology (Cooke, 2013; Eves, 1997). For example, the floods of the river Nile in Egypt and the induced need for redemarcation of the land estates, as well as the Babylonians' need to build large and famous technical works, promoted the invention or the discovery of Mathematics.

Mathematics, beyond any doubt, has its origin in the fifth, fourth and third millennium BCE. It began as a practical science, with the aim of assisting those ancient societies in engineering, agriculture, measurement, comparisons, commerce, and religious rituals (Bunt, Jones & Bedient, 1988; Eves, 1997). Mathematics grows like a plant from a seed, which sprouts and later continuously develops, producing roots, branches, leaves, flowers and fruits (Cooke, 2013).

Mathematics education, its teaching and learning, is, probably, as old as Mathematics itself. At the time when Mathematics was considered useful and gained “popular support”, the need for its transfer from one generation to the next arose, effortlessly. Mathematics education, in this sense of knowledge and skills transmission, is already observed in Egyptian, Mesopotamian, Indian and Chinese schools, although these were schools intended for “aristocratic students” (Damerow & Westbury, 1985). The first known systematic teaching of Mathematics, for the purpose of professional training of employees, has its origin in the third millennium BCE, in Mesopotamia (Schubring & Karp, 2014).

The formal history of mathematics education is an area that has an important legacy. The first academic projects in the relevant field began to appear, mostly in Germany, around the middle of the 19th century CE (Karp & Furinghetti, 2016; Schubring, 2014). However, it was only in the recent decades, especially since the 1980s onwards, that the interest was generalized (Goldin et al., 2016; Lerman, 1990; Kilpatrick 2020; 2014; 1992; Klein, 2003). In formal mathematics education which is a relatively new field (Stephan et al., 2015), as mentioned, significant progress has been made. However, despite the evolution, mathematics education is beset by serious problems, since, above all, it is difficult to motivate students and then keep their interest undiminished (Naeve & Nilsson, 2004; Schukajlow, Rakoczy & Pekrun, 2017). In fact, motivation has always been considered a unique precursor to success in Mathematics (Gottfried, 1985; Skaalvik, Federici & Klassen, 2015).

In this direction of improving the education of Mathematics, the motivation and the realization of the simplicity of Mathematics, in the last decades, due to the results of many studies and researches, the interest for exploring the role of the history of Mathematics in improvement of teaching and learning Mathematics has been stimulated (Barbin, Guillemette & Tzanakis, 2020; Barbin & Tzanakis, 2014; Clark et al., 2016; Clark, 2012). Aristotle, on the role and value of history, had stated disarmingly (Reich et al., 2007): “If you want to understand something well, then observe its beginning and its development”.

Moreover, an old Armenian proverb propagates historical knowledge: “He who lacks a sense of the past is doomed to live in the deep darkness of his own generation” (Gaither & Cavazos Gaither, 2012). Many great mathematicians have pointed out the importance of the history of Mathematics as a critical tool for clarifying “difficult" points” and for understanding the science of Thales and Archimedes (Furner & Brewer, 2016).

Understanding the new models in teaching and learning Mathematics, as well as the new pedagogical methods presupposes and requires some relevant knowledge of the historical past of Mathematics, given that the triptych “History, Mathematics and Education” (Karp & Furinghetti, 2016) are key, didactic and methodological axes, able to improve the daily teaching and learning process in current mathematics classes.

The fields of mathematics education and the history of Mathematics must interact, during the teaching and learning process, as, in this case, the opportunities and challenges are tempting, while the cognitive benefits can be significant, as well (Furinghetti, 1997; Radford et al., 2014; Tzanakis et al., 2002), although some objections have been raised (see e.g. Fried, 2001).

Despite some objections, it is assured that the integration and exploitation of the history of Mathematics in everyday teaching practice can be very beneficial not only to students but also for teachers, because history is strongly intertwined with the subject of Mathematics, mathematics education, as well as with the general intellectual perception of people, at any time and in any place (Ho, 2008; Panasuk & Horton, 2012; Shara, 2013).

The integration of the history of Mathematics into teaching can be exploited in two ways (Alpaslan, Isıksal & Haser, 2014; Jankvist, 2009; Tzanakis et al., 2002): a) history as a tool, associated with teaching of the courses, motivation, enhancement of mathematics knowledge and evolution of mathematical concepts and b) history as a goal, where subjects that are related to the development of mathematics science and its induced cultural load can be studied. The findings of many studies (Yee & Chapman, 2011) advocate the introduction of new mathematical ideas and concepts under the pedagogical cloak of a relevant historical context.

In general, something that concerns both the historical development of Mathematics and the way of human learning, every level of mathematical knowledge must be understood, as an improvement of lower-level knowledge and at the same time as a basis for higher-level knowledge (Wittmann, 2012).

Of course, such an “innovative” didactic move of utilizing history in Mathematics courses presupposes, necessarily, the readiness of teachers (Barbin & Tzanakis, 2014). There is no room for doubt, nor is there any need for argumentation as for as the established centrality of the role of teachers in the educational process and in the introduction of pedagogical innovations and reforms.

Mathematics teachers can act as decisive mediators in order to introduce and integrate into education reforms, innovations and modifications into the Curricula (Furinghetti, 1997; Prestridge & Main, 2018; Serdyukov, 2017). In fact, it has been found (Russo & Hopkins, 2019; Gilbert et al., 2014; Adamson et al., 2003) that the students of those teachers who apply innovative teaching practices and adopt reforms, achieve not only higher grades and better performance in various tests, but also gain more learning benefits.

Teachers' attitudes and perceptions, which are now a visible educational variable, are strongly linked to teachers' views mathematics teaching and learning and are crucial for shaping interventions during daily teaching practice in classrooms (Maasz & Schlöglmann, 2019; Schoen & LaVenia, 2019). However, and in continuation of the above, beliefs and attitudes can be changed (obviously, for the better), through the study and exploitation of historical facts of Mathematics (Alpaslan, Isıksal & Haser, 2014; Charalambous, Panaoura & Philippou, 2009). Knowledge of the history of Mathematics helps teachers to gain self-confidence and self-esteem, to develop their mathematical knowledge (conceptual and algorithmic) and improve their didactic readiness, during the lessons. Knowledge of the history of Mathematics enables teachers to understand and identify the simple ideas and concepts on which Mathematics is based.

The present paper, in the context of improving the teaching and learning of algorithms, utilizing the important role of the history of Mathematics in improving mathematics teaching and learning, presents and studies two ancient algorithms, which exploit errors and solve almost all the word problems in Mathematics. By studying many ancient sources, as well as the suggestions of modern researchers and scholars, a journey of 4,000 years is recorded, until today, through the development of the algorithmic knowledge, from the early times of the ancient Egyptians and Babylonians. This work, also, presents several ways through which students could benefit from participating in a relevant school project.

The method (algorithm) of simple and double false position

In the constant struggle to find solutions to problems, many simple and complex methods of solving have been proposed, since the archaic years, since the time of primitive civilizations. In one of them, the solution of nonroutine (unusual) and well-defined problems is reduced to a simple, procedural, routine task, through the activation of an easy algorithmic solution method. This method can be taught and applied, easily, even by students of the upper grades of Primary School.

Indeed, there is a "trial and error" method in Mathematics, titled "false position" ("false assumption" or "regula falsi"), which solves the linear word problems resulting in a first-degree equation. The false position method (algorithm) is based on conjectures and errors, the simple finding of which, through the well-known methodology associated with direct proportions and equal ratios, gives the solution easily. Its advantage is the absence of algebraic symbolism. This analytical, non-algebraic technique was widely used, until the 16th century, to solve related problems (Dubbey, 1975; Høyrup, 2013).

From the 17th century onwards, the management of such linear problems became quite easy, through the invention of new algebraic methods. The development of algebraic symbolism and the ease of formulating and solving first-order equations (Sanford, 1951), dealing with related problems, contributed to the false position methods (simple & double one) being sidelined.

The simple false position method in ancient Egypt and Babylonia

The simple false position method was found in ancient Egyptian papyri written in hieratic script (a type of Hieroglyphs) and cuneiform tablets from Ancient Babylonia (Easton, 1967; Chabert, 1999; Chakerian, 2004; Høyrup, 2013; Imhausen, 2016; Katz, 2009). Indeed in Babylonian tablets (without, however, being absolutely certain about the type of algorithm Babylonian used) and on Rhind papyrus (or Ahmes) there are the first applications of the so-called "simple false position" method (Dubbey, 1975; Chabert, 1999; Cajori, 1991; Høyrup, 1994; Folkerts & Hughes, 2016; Imhausen, 2007; Katona & Szűcs, 2017; Katz, 2009). In the Egyptian papyrus, in particular in the problems 24 to 40, the simple false position is the only suggested method of solving (Bunt, Jones & Bedient, 1988; Chace, Manning & Archibald, 1927; Eves, 1958). In the 24th problem from Rhind papyrus for example, a quantity was sought, in which, when its one seventh added together with the quantity, the number 19 was obtained. The 31st problem was the following: "A quantity and its 2/3, its 1/2 and its 1/7 added together become 33. What is the original quantity?".

A first degree equation, of course, makes the solution much easier and simpler. A modern solution via equation and algebraic symbolism follows.

Let be x the original quantity. Then the equation is:

$$x+\frac{2}{3}\cdot x+\frac{1}{2}\cdot x+\frac{1}{7}\cdot x=33 \Rightarrow \left(\frac{42}{42}+\frac{28}{42}+\frac{21}{42}+\frac{6}{42}\right)\cdot x=\frac{1.386}{42} \Rightarrow 97\cdot x=1.386 \Rightarrow x=\frac{1.386}{97}=14\frac{28}{97}$$

Without the use of equations, these word problems demand high intellectual commitment and close mental attention (Eves, 1983), since the argumentation is based on purely rhetorical reason. With methods of rhetorical argumentation a solution, for the above 31st problem from Rhind papyrus could be as follows: "2/3, 1/2 and 1/7, i.e. 55/42 of the quantity, added with the quantity itself (42/42) give a sum of 97/42. Therefore, since 97/42 of the original quantity is equal to 33, the original quantity will be equal to 33:(97/42) = 1386/97. However, students (as well as adults) face many difficulties, when solving such word problems (Caldwell & Goldin, 1979; Daroczy et al., 2015; Salemeh & Etchells, 2016).

The ancient Egyptians, according to what is recorded in the Rhind papyrus, solved the problem as follows:

- Suppose the original quantity is 42 (position/assumption - arbitrary, incorrect solution).
- Then, 42+42·(2/3)+42·(1/2)+42·(1/7) = 42+28+21+6 = 97 (error, because the correct answer is 33).
- According to the version of the Egyptian and Babylonian simple false position method, to obtain the right answer (33), the wrong result (97) of the trial must be multiplied by the fraction 33/97 [97·(33/97) = 33]. Proportionally, doing the same, the original position/assumption (42) must be multiplied by 33/97, in order to obtain the requested right answer [42 · (33/97) = 1386/97], as we have also previously calculated, via an equation.

In fact, the solution from the Rhind Papyrus, although it follows exactly this algorithmic procedure, is presented very complex. The reason can be found in the fractions representation by the ancient Egyptians, since they used only unit fractions, to represent the fractions. It is believed (Clagett, 1999) that in some problems of the ancient Egyptians, the original solutions of simple false position were not included in the papyri, so some authors were forced to recreate the solutions.

It can easily be seen that the simple false position requires one trial and solves all the problems that result in first degree equations with no constant term (in the form a·x = b). It is based on the equality a·(d·x) = a·(d·x) = d·b (where d·x is the trial), i.e. the change in the input of the algorithm is proportional to the change of its output. Otherwise, if a·x = b and x_1 is the arbitrary, incorrect original value (position/ assumption) for x, then $a \cdot x_1 = b_1$ (b_1 is the error) and, obviously, the following ratio is applied:

$$\frac{x}{x_1} = \frac{b}{b_1} \Rightarrow x = \frac{x_1 \cdot b}{b_1}$$

The exploitation of the above ratio gives easily the correct solution:

$x = \frac{42 \cdot 33}{97} = \frac{1.386}{97}$, where $x_1 = 42$ is the arbitrary, incorrect original value, b=33 and b_1= 97 is the error.

This ratio explains more supervisoryly and convincingly the simple false position method, as a version, in fact, of the rule of three. In addition, the simple false position method is equivalent to the construction of a straight line equation (linear interpolation), if one point is known [the other one is the point (0, 0) and in this case, the line passing through the origin], because it requires one trial.

Simple false position methods were also used by the ancient Babylonians, although any attempt to decipher a general algorithm they used is uncertain. Bell (2012) considers (rather justifiably) that Babylonian Mathematics was much more advanced than the Egyptian Mathematics and superior as well. In general, Egyptians and Babylonians used almost the same algorithms in their Mathematics, although the problems of the Babylonians, which they solved, through simple false position are more complex and complicated (Hannah, 2007; Høyrup, 2013). However, since the two civilizations developed in the same period of time, it becomes difficult to find out which civilization (perhaps) was influenced and copied the other (Papakonstantinou, 2009; Rudman, 2010). In any case, there are astonishing and unexpected links and similarities between Egyptian and Babylonian Mathematics (Friberg, 2005).

For example, the following problem (which is very similar to those of the ancient Egyptians) comes from the Babylonian tablet YBC 4669 (Chabert, 1999): "*I have consumed two thirds of my provisions and I have 7 left. What was the original amount of my provisions*?". In this problem an incorrect answer was recorded on the tablet, while as for as another similar problem, where the weight of a stone was requested, the recorded solution is too vague and confusing, so it is almost impossible to identify the algorithm used.

The simple false position method in Medieval India

In India, the only clear reference to the simple false position method can be found in the very popular work "Līlāvatī" by the greatest mathematician in Medieval India Bhāskarācārya (1114–1185), also known as Bhāskara II (Gupta, 2016; Papakonstantinou & Tapia, 2013; Sarton, 1931; Smith, 1958; Van Brummelen, 2016).

In his work "Līlāvatī" (probably his daughter's name), which included chapters on Arithmetic (e.g. rule of three but also an early version of the current multiplication algorithm) and Computational Geometry, Bhāskara II, who was greatly influenced by the works of the ancient Chinese (Swetz & Kao, 1988) separated the rules of ratios from the numerical calculations and created a later relevant chapter, which he called prakīrnaka ("miscellaneous rules").

The "miscellaneous rules", which was the third chapter in Līlāvatī, included rules for inverse operations, summaries and differences, root calculations, formulas in combinatorics and the "Ista-Karma" method. "Ista-Karma", meaning calculation with a given number, was the Indian name that Bhāskara II gave to the simple false position. The method, as described by the great Indian mathematician, was exactly the following (Puttaswamy, 2012): "*Whenever a required number is multiplied or divided by whatever fraction of the number is found to have been increased or diminished, assume an optional number; on it perform the same operations in accordance with the statement of the problem; multiply the given number in the statement of the problem by the assumed number and then divide this product by the number which resulted from the above operation. The quotient will be the required number.*"

The simple false position method, through the perspective of Bhāskara II, is interpreted and explained as follows: Let x be the required number and a the fraction or sum of fractions. Then, $a \cdot x = b$ (1), where b is the given number. If x_1 is the optional number, then $a \cdot x_1 = b_1$ (2), where b_1 is the false calculated number. Dividing (1) by (2), it gives $\frac{a \cdot x}{a \cdot x_1} = \frac{\beta}{\beta_1} \Rightarrow \frac{x}{x_1} = \frac{\beta}{\beta_1} \Rightarrow x = \frac{x_1 \cdot \beta}{\beta_1}$, according, exactly, to the instructions of Bhāskara II.

A typical problem (slightly adapted) from the "Līlāvatī" is the following (Sarmaa & Zamanib, 2019): "A pilgrim gave away half of his money (gold coins) at Prayāga; two-ninths of the remainder at Kāśī; one-fourth of the remaining amount for the tax on the road; six-tenths of what remained at Gayā; he returned home with the remaining sixty-three gold coins. Tell quickly the original amount of gold coins of the pilgrim".

Assuming the pilgrim had 200 gold coins, then he gave away 100 gold coins at Prayāga; (200-100)·2/9 = 200/9 gold coins at Kāśī; [200-100-(200/9)]·1/4 = 700/36 gold coins for the tax on the road; [200-100-(200/9)-(700/36)]·6/10 = 35 gold coins at Gayā. Therefore the pilgrim returned home with the remaining 200-[100+(200/9)+(700/36)+35] = $200 - 176\frac{2}{3} = 23\frac{1}{3}$ gold coins. According to Bhāskara II, the original amount of gold coins of the pilgrim is $\frac{200 \cdot 63}{23\frac{1}{3}} = 540$ gold coins.

The double false position method in ancient China

The Chinese book "Jiuzhang Suanshu" ("Nine chapters on the mathematical art"), which can be considered equivalent to Euclid's Elements, contains Mathematics of the Chinese, since 1000 BCE to 50 CE. Sarton (1927) determines this period up to the 27th century BCE. "Jiuzhang Suanshu", written between 100 BCE and 100 CE (Lam, 1987), includes 246 real life problems, in the form of "question-answer-explanation", without, of course, proof (Boyer, 1968; He, 2002; Merzbach & Boyer, 2011), since proofs were introduced in Mathematics, during the era of Thales of Miletus and later of Pythagoras (Bell, 1986). The proof derived from Chinese seems to have been the result of complex interactions (Martzloff, 1994), among the various belief systems and spiritual currents that dominated the Chinese world, mainly Confucianism, Sophism and Taoism.

The 7th chapter of "Jiuzhang Suanshu", titled "Ying Bu Zu" (盈不足), includes problems solved by the algorithm of double false position. Indeed, in this chapter are the beginnings of applying the double false position method (Yuan, 2012; Chemla, 1997; Eves, 1990; Swetz, 1972). There seems to be no historical doubt that the double false position method was not used in either Egypt or Babylon (Høyrup, 2013), although there are erroneous views on this (e.g. Papakonstantinou & Tapia, 2013; Papakonstantinou, 2009). In fact, the Chinese was the first civilization to devise this form of solving first degree equations or systems of linear equations in two variables (with 2 unknowns), by setting two arbitrary original values.

The double false position requires two trials and solves all the problems that result in first degree equations with a constant term (in the form $a{\cdot}x+c = b$). If the two errors correspond to values, which are either bigger or less than the correct solution, then they called "similar" ones, while if the errors are correspond to values, one of which is larger and the other smaller than the solution, then they called "dissimilar" ones. If a value of an error is bigger than the correct solution, then there is an excedent (surplus, excess), while if a value of an error is less than the correct solution, then there is a deficit. This process called "Excedent and Deficit" (Ying Bu Zu), according to Zhang & Qin (2011), is the oldest method for solving algebraic equations.

In particular, the chapter "Ying Bu Zu" includes twenty (20) problems. The first eight problems study excedents (surpluses, excesses) (Ying) and deficits (Bu Zu) and are divided into three types (Lam, 1994): (a) with only two excedents or only two deficits; (b) with one excedent and one deficit; and (c) with one excedent and the exact quantity or one deficit and the exact quantity (i.e. nor excedent neither deficit). The first type refers, obviously, to similar errors, the second one to the dissimilar errors, while the third one can be included in either of the previous two. The remaining 12 problems of the7th chapter of "Jiuzhang Suanshu" concern only problems with dissimilar errors (i.e. there is an excedent and one deficit). For unknown reasons, the other two types of problems were not further studied by the ancient Chinese.

The double false position is equivalent to the construction of a straight line equation (linear interpolation), if two point are known, because it requires two trials. If there are two excedents or 2 deficits the solution is: $\frac{x_1 \cdot e_2 - x_2 \cdot e_1}{e_2 - e_1}$, while if there are one excedent and one deficit the solution is: $\frac{x_1 \cdot e_2 + x_2 \cdot e_1}{e_2 + e_1}$, where: x_1 is the first arbitrary value, x_2 is the second arbitrary value, e_1 is the first error and e_2 is the second error.

It can be easily ascertained if the errors are "similar", the difference of the products ($x_1{\cdot}e_2$ and $x_2{\cdot}e_1$) is divided by the difference of the errors, while if the errors are "dissimilar", the sum of the products ($x_1{\cdot}e_2$ and $x_2{\cdot}e_1$) is divided by the sum of the errors. In both cases the quotient is the requested answer.

Let x_1 and x_2 be the input values of x, during the trials, and b_1 and b_2 the corresponding results, then $a{\cdot}x_1+c = b_1$ and $a{\cdot}x_2+c = b_2$. Since $e_1 = b_1-b$ and $e_2 = b_2-b$, if $b_1, b_2 <b$, then e_1 and e_2 are positive numbers as excedents, while respectively, if $b_1, b_2 >b$, e_1 and e_2 are negative numbers as deficits. The value of x, of course, does not differ, when the errors have the same sign. However, in the case of dissimilar errors, without loss of generality, assume that $e_1 < 0$ and $e_2 > 0$. Then, $x = \dfrac{x_1 \cdot e_2 - x_2 \cdot (-e_1)}{e_2 - (-e_1)} = \dfrac{x_1 \cdot e_2 + x_2 \cdot e_1}{e_2 + e_1}$, since during the application of the algorithm positive numbers are always placed (Chakerian, 2004; Easton, 1967; He, 2002; Papakonstantinou, 2009).

In a variation of the double false position, as presented in the very popular American textbook of the 19th century “Daboll's School- master's Assistant” (Daboll, 1817), the first trial is multiplied by the second error and the second trial by the first error. This way (as if numbers arranging in a grid) was attributed from the 13th century onwards with the symbol X (Cajori, 2007), and, for this reason, began to be considered as the sign of multiplication.

The methodology of double false position will be clarified, through two examples. Let's consider the following problem: “There are rabbits and chickens on a small farm. If the total number of animals is 35 and their legs are 96, find how many animals of each species are”. For the solution, two excedents will be exploited.

Suppose the chickens are $15 = x_1$ (hence rabbits are 20).

- 1st trial: 15·2+20·4 = 110 (legs).
- 1st error: e_1 = 110-96 = 14 (excedent).

Suppose the chickens are $10 = x_2$ (hence rabbits are 25)

- 2nd trial: 10·2+25·4 = 120 (legs)
- 2nd error: e_2 = 120 - 96 = 24 (excedent).
- Difference of errors: $e_2 – e_1$ = 24-14 = 10

We place the trials and errors in this way:

trials: 15 10
errors: 14 24

Thereafter making the cross-multiplications and using the relevant type $\dfrac{x_1 \cdot e_2 - x_2 \cdot e_1}{e_2 - e_1}$, we can find:

15·24 - 10·14 = 360 - 140 = 220. Consequently, the chickens are 220:10 = 22 and the rabbits are 13.

Now for the solution, one expedient and one deficit will be exploited. Suppose the chickens are $10 = x_1$ (hence rabbits are 25).

- 1st trial: 10·2+25·4 = 120 (legs).
- 1st error: e_1 = 120-96 = 24 (excedent).

Suppose the chickens are $25 = x_2$ (hence rabbits are 10)

- 2nd trial: 25·2+10·4 = 90 (legs)
- 2nd error: e_2 = 90 - 96 = (-)6 (deficit).
- Sum of errors: $e_2 + e_1$ = 24+6 = 30

We place the trials and errors in this way

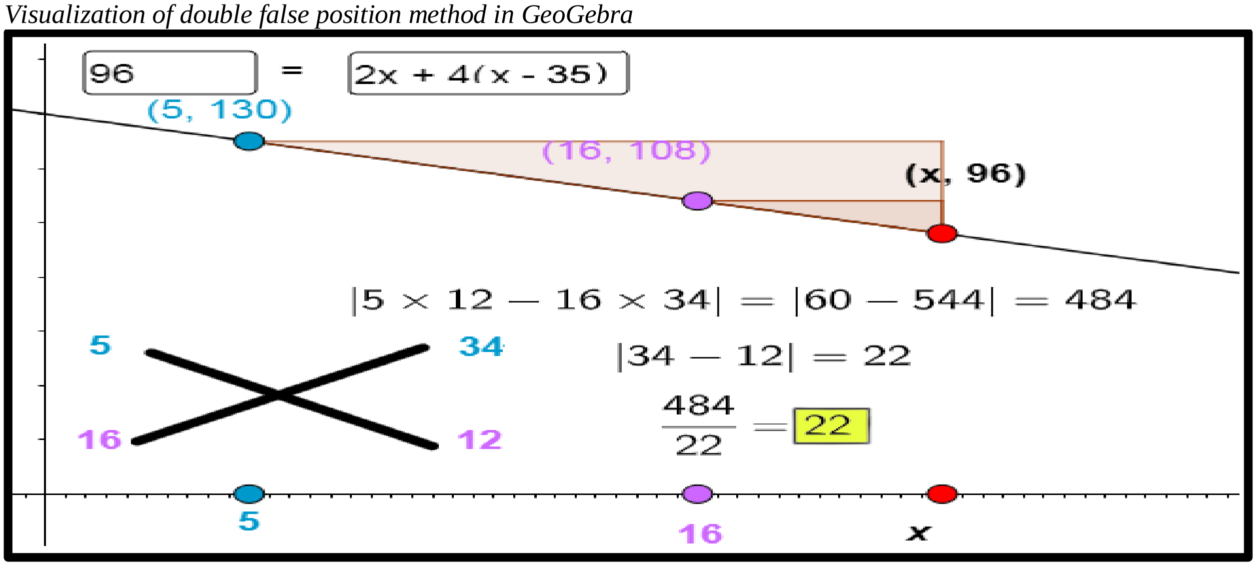
trials: 10 25
errors: 24 6

Thereafter making the cross-multiplications and using the relevant type $\frac{x_1 \cdot e_2 + x_2 \cdot e_1}{e_2 + e_1}$, we can find:

10·6 + 25·24 = 60 + 600 = 660. Consequently, the chickens are 660:30 = 22 and the rabbits are 13.

For a better understanding, the false position method could be visualized using Dynamic Geometry Software (e.g. GeoGebra). In figure 1, below, there is the solution of the above problem, where the original values are $x_1 = 5$, $x_2 = 16$, $b_1 = 130$, $b_2 = 108$, the corresponding errors are $e_1 = 34$, $e_2 = 12$ and the difference of errors is (there are two excedents): $e_1 - e_2 = 34\text{-}12 = 22$.

Figure 1
Visualization of double false position method in GeoGebra



Indeed, through this interactive representation - visualization it is possible for students to realize that the false position method is equivalent to the construction of a straight line equation (linear interpolation), where the two known points are (x_1, b_1) and (x_1, b_1) in the case of the double false position method, while in the case of simple one the known points are (x_1, b_1) and (0, 0).

The false position methodological algorithm took its general (known) form, much later, from the Arabs and Europeans (Katz, 2009). Arabs used the algorithm of double false position, under the name “hisab al-khata'ayn” (Devlin, 2012; Oaks, 2012; Schwartz, 2011), which meant “reckoning by two errors”. The great Moroccan mathematician Ibn al-Bannāʾ al-Marrākushī named a version of double false position method as kiffāt- method of the scales (Fink, 1900; Schwart, 2011), due to the created shape that looks like a scale, during the construction of the solution.

Leonardo of Pisa (Fibonacci) was not only a devotee of false position method, but also introduced the method in Europe. In his classical book “Liber Abaci” (12th and 13th chapter), that published in 1202 CE, Leonardo of Pisa presented various methods of solving mathematical enigmas and problems. One of these solutions (in third section of Chapter 12) was the false position method. Leonardo of Pisa called false position as the method of trees, because in the first example he used to study and clarify the method, he asked for the length of a tree (Hannah, 2007; Sigler, 2002): “There is a tree of which 1/3 and 1/4 lie under the ground. If the part of the tree under the ground is 21 meters, what is the total length of that tree?”.

In the next seven centuries after Fibonacci, the false position method, although scientifically marginalized in the 17th century CE, due to the development of symbolic methods, it was still part of the textbooks in many Asian and American educational systems, until the beginning of the 20th century (Daboll, 1817; Karp & Schubring, 2014; Katz & Hunger Parshall, 2014).

False position method and its potential learning implications in schools

The simple-simplistic (methodological) algorithm of the false position covers a period of 4,000 years, from the beginnings of the recorded civilization, reaching (latently) to one of its most important peaks, the differential calculus. This method has a timeless and insistent presence in mathematical production and literature. The simple but effective algorithm of the false position intersecting almost all pre-Christian and pre-medieval civilizations, has reached through Fibonacci in Europe. Historically, the false position, as a methodological algorithm, along with the multiplication of ancient Egyptians is the oldest recorded algorithm.

The false position method has created challenging learning conditions around the world for African, Asian, Arab, European and American students. Today, primary and middle school students are able, through false position method, to sense the timelessness of algorithmic processes as the main mathematical knowledge. Moreover, a historical retrospection of false position method from the depths of history, as the subject of a project, can unfold the history of Mathematics, from its timid beginning to its exciting current evolution.

The false position method is a magnetic, attractive technique, ideal for teaching and learning proportions, ratios and the rule of three, as well as for highlighting the causes of prevalence of the modern methods over older ones. Moreover, the fascination of the history of Mathematics and the simple starting points of ideas that support the scientific edifice of Mathematics can be some attractive teaching parameters and modules of the project.

Through this project, students can study and know parts of Egyptian, Babylonian, Arabic and Chinese Mathematics. For example, students may contact with Egyptian Mathematics and its additive number system. In additive number systems, the numerals (symbols) are combined in order to represent numbers. Numbers are the sum of the values of these numerals. Ancient Egyptians used a decimal and non-positional numeral system. Its hieroglyphic alphabet consisted of seven different symbols (Bunt, Jones & Bedient, 1988; Cajori, 1991; Imhausen, 2016; Katz, 2009). Each symbol denoted powers of 10, from one unit up to one million (10^0, 10^1, 10^2, … , 10^6). Figure 2, below, shows the seven hieroglyphic symbols of ancient Egyptian number system.

Figure 2
Ancient Egyptian numerals

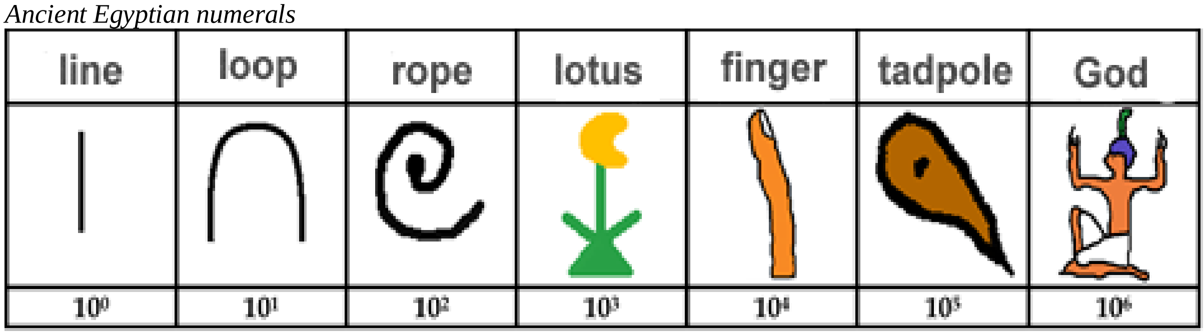
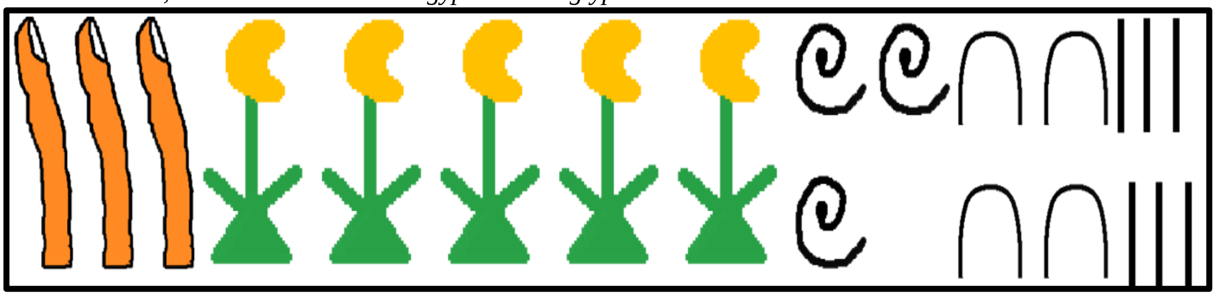


Figure 3, below, shows the number 35,346 written in ancient Egyptian hieroglyphs. The system was additive, therefore the order of the symbols does not matter.

Figure 3
The number 35,346 written in ancient Egyptian hieroglyphs



Also, the false position method can be part of the relevant project on the collection of effective timeless methods of problem solving, invented by man. In particular, the Arabs' method of the scales can be a motivator for search, over the centuries, for tricks and stratagems that support memorization and learning (e.g. alternative strategies, rhymes, visualizations, calculating machines etc.).

Moreover, students can understand the pedagogical value of error and realize that they can learn from their mistakes. Many popular textbooks introduced the students of that time to the secrets of the “paradox” produced and brought to the fore by the unexpected exploitation of mistakes (errors) in the educational process and problem solving. According to Zhuang Zhou (Zhuangzi), a great Chinese philosopher, “*everyone knows the usefulness of the useful, but no one knows the usefulness of the useless*” (Goodman, 2016). As for errors (which inevitably occur in Mathematics) the American author Walter Lippmann (1889-1974) gave a general, useful dimension to them, after pointing out that “*the study of error is not only in the highest degree prophylactic, but it serves as a stimulating introduction to the study of truth*”. The French poet Louis Aragon (1897-1982), also, stated that “*error is certainty’s constant companion. Error is the corollary of evidence. And anything said about truth may equally well be said about error: the delusion will be no greater*” (Thomsett & Thomsett, 2015).

Finally, studying the false position method, students can learn that this method is responsible for the adoption of the symbol “X” for multiplication. It was Leonardo of Pisa who introduced the symbol X (Cajori, 2007) in his important book Liber Abaci. The false position method is responsible, as well, for the introduction of the signs “-” and “+” in Mathematics.

There is a great certainty that the need for different representations of dissimilar errors was the reason for the introduction of the minus and plus sign in Mathematics. In the book of the German mathematician Johann Widman (1489-1526), titled “Mercantile Arithmetic or Behende und hüpsche Rechenung auff allen Kauffmanschafft”, published in 1489 in Leipzig, the signs for positive and negative numbers first appeared (Cajori, 2007). It is strongly believed (Cajori, 1991; Needham & Ling, 2005; Sanford, 1951) that Widman invented the signed numbers because he wanted to familiarize his students with the use of the double false position method, especially in the case of dissimilar errors.

General Conclusion

Undoubtedly, a historical look at Mathematics, as a teaching tool, increases the motivation for learning, stimulates multicultural approaches and upgrades students to little researchers (Fauvel, 1991). A historical research in Mathematics can create channels and preconditions to highlight its cultural brilliance, enhance the motivation for studying and learning and consequently facilitate its understanding. The simple ideas with which Mathematics is "woven", it is possible, through a reverse course to the past, to be perceived, a fact that, certainly, can contribute to the demystification of their difficulty. After all, the declaration of the great French philosopher Auguste Comte (1798-1857) is characteristic and corroborating: "*To understand a science, you must know its history*" (Chowdhury, 2014). Indeed, Mathematics, without knowledge of its historical origins, is stripped of its cultural grandeur. Undoubtedly, this greatness is due to the fact that Mathematics is a human creation (Simmons, 2017).

Through the algorithm of false position, the value of error as a basis and means for solving problems has evolved certainly into a decisive pedagogical and social principle over the centuries. Furthermore, in this light of the false position method, this great pedagogical value of error can also be validated in the search for the solution of many mathematical and non-mathematical problems. "*Error is often more serious than truth*" (Dingle, 2001), as Benjamin Disraeli (1804-1881) said, but a phrase attributed to Victor Hugo (1802-1885) includes a ... very serious truth (Cameron, 2014): "*To rise from error to truth is rare and beautiful*".

Further research could examine other algorithms since ancient times, such as the algorithm of multiplication (Russian Peasant Multiplication) and division of the ancient Egyptians, the methods of multiplication and division in Vedic Mathematics or Diophantus' innovative, ahead of his time problem solving methods. Alternative calculational and methodological algorithms could also be studied, such as several algorithms, around the world, for the four arithmetic operations, the Karatsuba algorithm, the Euclidean algorithm and Gauss's intelligent algorithm for calculating the sum of the first 100 integers.

Moreover, a project, that could be possibly very interesting, from a learning perspective, is about the history of algorithmic knowledge, over the centuries, up to their current «scientific omnipotence». The story of the great Arab mathematician Al Khwarizmi (the word algorithm comes from his name) would be an important aspect of another project. Finally, a goal of a future research and study could be an investigation of teachers' views, regarding the possibility of integrating ancient and alternative algorithms in the Mathematics Curriculum of Primary and Secondary Education.

In any case, there is a high probability through these educational tasks and interventions that students will realize that algorithms are a basic and critical mathematical knowledge, which in coexistence with conceptual knowledge, metacognition and reasoning processes is possible to create a powerful didactic kit in order to reveal the richness, variety, simplicity and beauty of Mathematics.

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